

REGULAR SEMIGROUPS SATISFYING CERTAIN CONDITIONS ON IDEMPOTENTS AND IDEALS

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ABSTRACT. The structure of regular semigroups is studied (1) whose poset of idempotents is required to be a tree or to satisfy a weaker condition concerning the behavior of idempotents in different $\mathfrak{D}$ -classes, or (2) all of whose ideals are categorical or satisfy a variation thereof. For this purpose the notions of D -majorization of idempotents, where D is a $\mathfrak{D}$ -class, $\mathfrak{D}$ -majorization, $\mathfrak{D}$ -categorical ideals, and completely semisimple semigroups without contractions are introduced and several connections among them are established. Some theorems due to G. Lallement concerning subdirect products and completely regular semigroups and certain results of the author concerning completely semisimple inverse semigroups are either improved or generalized.

1. Introduction. An arbitrary regular semigroup S has at least two outstanding features: (1) may have a very complicated structure, (2) abounds in idempotents. For these two reasons, it is natural to impose suitable restrictions on the partially ordered set E_S of idempotents of S in order to obtain various classes of regular semigroups more susceptible to a treatment which might eventually lead to a complete determination of its structure. Another type of restriction is a statement concerning different kinds of ideals of S . Such restrictions are sometimes expressed by means of Green's relations. As long as these restrictions are very strong, they lead to a complete determination of the structure (mod, e.g., semilattices or groups) of the semigroup satisfying them, witness: completely 0-simple semigroups (see [1], [9]), trees of completely 0-simple semigroups [8], simple regular ω -semigroups ([5], [10]), etc.

In addition to the frequent hypothesis that our semigroup be completely semisimple, we impose restrictions of either type mentioned above with the following variations: (1) the partially ordered set E_S is required to be a tree or to satisfy a condition concerning the behavior of idempotents in different $\mathfrak{D}$ -classes of S , (2) all ideals are required to be categorical or satisfy a modified version thereof involving $\mathfrak{D}$ -classes of S . With suitable restrictions on a regular semigroup, we are able to establish several connections among these conditions, and in some instances, characterize the semigroups in question in terms of subdirect products

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of (general or special) completely 0-simple semigroups, or give their complete structure. The frequent change of hypotheses strongly suggests that many of our results are not the best possible. This opens a great number of problems particularly concerning subdirect products of 0-bisimple regular semigroups.

Summary. We begin in §2 by considering the relationship of two regular $\mathcal{D}$ -classes A and B of an arbitrary semigroup S in terms of the behavior of their idempotents and functions from B into A satisfying certain multiplicative conditions. The principal idea here is a generalization of the development first expounded by Lallement [6, Chapitre II, §3] for the case of a completely semisimple semigroup. We introduce two binary relations on the set of regular $\mathcal{D}$ -classes of S and the notions of C - and σ -majorization, which are then used throughout the paper. Making a free use of the results in the author's paper [14], the main result of this section is a connection between the function χ_A introduced in [14] and A -majorization (a condition on idempotents) for a regular semigroup S . A number of lemmas are stated in a general setting and seem to be of independent interest.

In §3 we specialize the situation by further restricting the condition on the functions χ_D for a regular semigroup S , and deduce a slight improvement of a theorem due to Lallement [6, 2:12, 2.17] concerning subdirect products of completely 0-simple semigroups, giving a simpler proof. This is further specialized in §4 by considering completely regular semigroups, the main hypothesis always being $\mathcal{D}$ -majorization. Again we improve a theorem of Lallement [6, 2.21–2.23], giving several characterizations of the class of completely regular semigroups satisfying $\mathcal{D}$ -majorization.

We introduce in §5 the concept of a σ -categorical ideal, and for a certain class of regular semigroups S show that $\mathcal{D}$ -majorization implies that all ideals of S are $\mathcal{D}$ -categorical and that if E_S is also a tree then all ideals are categorical. In order to establish a partial converse of these statements, we consider in §6 a more general situation. Indeed, we introduce the notion of a completely semisimple semigroup without contractions and give simple necessary and sufficient conditions in order that an extension V of a primitive regular semigroup S without contractions be determined by a partial homomorphism. This is used in §7 to obtain a partial converse of some statements in §5. Several results in §§5–7 have been established for inverse semigroups in the author's paper [15].

Notation. We follow the notation and terminology used in [1] with a few minor exceptions. The following is a list of frequently used notation. If A is a non-empty subset of a semigroup S , then E_A denotes the partially ordered set of idempotents contained in A , where $e \leq f \Leftrightarrow e = ef = fe$. To avoid cumbersome exceptions, we often suppose that the semigroup S under consideration has a zero, which is denoted by $S = S^0$; usually an obvious modification yields the desired statement for the case without zero and will not be explicitly stated. For

sets A and B , we write $A \setminus B = \{a \in A \mid a \notin B\}$, and do not distinguish between an element and the set consisting of that element alone. The set of nonzero elements of a semigroup S with zero is denoted by S^* . If D is a $\mathcal{D}$ -class of S , then D^0 denotes the groupoid defined on the set $D \cup 0$ where $0 \notin D$, 0 acts as the zero, and all products of elements of D that, in S , fall outside of D are set equal to 0 . If α is a homomorphism of S , $\ker \alpha$ is the congruence on S induced by α . We use the following notation frequently:

$$M_e = \{f \in E_S \mid f \leq e\} \quad \text{where } e \in E_S.$$

2. **Conditions on idempotents of regular $\mathcal{D}$ -classes.** We will establish here a number of auxiliary statements mainly concerning two regular $\mathcal{D}$ -classes of an arbitrary semigroup. These results are fundamental for a large portion of the paper.

Throughout this section, A and B denote regular $\mathcal{D}$ -classes of an arbitrary semigroup $S = S^0$.

Lemma 2.1. *If $(AB \cup BA) \cap A \neq \emptyset$, then there exist $e \in E_A$ and $f \in E_B$ such that $e \leq f$.*

Proof. Let $a \in A$, $b \in B$ and suppose that $ab \in A$. Let $f \in E_B$ be such that $b = bf$, and let $x = ab$, x' be an inverse of x , $e = fx'x$. Then

$$e^2 = (fx'x)(fx'x) = fx'(xf)x'x = fx'xx'x = fx'x = e,$$

$$ef = (fx'x)f = fx'(xf) = fx'x = e = fe,$$

so that $e \leq f$. Further,

$$x = x(x'x) = abx'x = a(bf)x'x = ab(fx'x) = xe$$

which together with $e = (f'x')x$ implies that $e \mathcal{L} x$ and thus $e \in A$. The case $ba \in A$ is analogous.

It follows from [2, Théorème 2] that under the hypothesis of the lemma, for any $g \in E_B$, we have $M_f \cap E_A \cong M_g \cap E_A$ as partially ordered sets. We thus introduce the following notation:

$A \leq B$ means that for some (and thus any) $f \in E_B$, $M_f \cap E_A \neq \emptyset$,

$A \leq B$ means that for some (and thus any) $f \in E_B$, the poset $M_f \cap E_A$ has a greatest element, to be denoted by $\bar{f}$.

The relations $\leq$ and $\leq$ are defined on the set Δ of all regular $\mathcal{D}$ -classes of S . It follows from the definition that both $\leq$ and $\leq$ are reflexive and that $\leq$ is also transitive. In a simple regular ω -semigroup with $\mathcal{D}$ -classes $D_1, D_2, \dots, D_n$ where $n > 1$, we have $D_i \leq D_j$ for any $1 \leq i, j \leq n$ showing that $\leq$ and thus also $\leq$ need not be antisymmetric, see [5] or [10]. In fact, $\leq$ need not even be transitive. For take the set $T = \{1, 2, 3\}$, and transformations $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$,

$\gamma = \begin{pmatrix} 123 \\ 111 \end{pmatrix}$, $\delta = \begin{pmatrix} 123 \\ 222 \end{pmatrix}$; then $D_\gamma = D_\delta \leq D_\beta$, $D_\beta \leq D_\alpha$, but $D_\delta \not\leq D_\alpha$.

We will have many occasions to consider various functions from B into A which have certain multiplicative properties. It is then convenient to introduce the following notation. Let $\phi: B \rightarrow A$ be a function and $a \in A$, $b, b' \in B$ be arbitrary elements. We will say that ϕ satisfies condition (α) , resp. (β) , etc., if

$$(\alpha) \quad ab \in A \Rightarrow ab = a(b\phi), \quad ba \in A \Rightarrow ba = (b\phi)a;$$

$$(\beta) \quad E_B\phi \subseteq E_A;$$

$$(\gamma) \quad ab \in A \text{ or } a(b\phi) \in A \Rightarrow ab = a(b\phi), \quad ba \in A \text{ or } (b\phi)a \in A \Rightarrow ba = (b\phi)a;$$

$$(\delta) \quad bb' \in B \Rightarrow (bb')\phi = (b\phi)b' = b(b'\phi) = (b\phi)(b'\phi);$$

and if C is a regular $\mathcal{D}$ -class of S and $\psi: C \rightarrow A$ (for $C = B$, take $\psi = \phi$),

$$(\epsilon) \quad b \in B, \quad c \in C, \quad bc \in A \Rightarrow bc = (b\phi)c = b(c\psi) = (b\phi)(c\psi).$$

The purpose of the next two lemmas is to construct a function from B to A satisfying (α) and (β) provided that $A \leq B$.

Lemma 2.2. *Let $A \leq B$, $b \in B$, and $e, f \in E_B$ be such that $e \mathcal{R} b$, $b \mathcal{L} f$. Then $\bar{e}b = b\bar{f} \in A$ and this element is independent of the choice of $e \in R_b$, $f \in L_b$.*

Proof. Recall that $\bar{e}$ is the greatest element of $M_e \cap E_A$. To prove the first statement, by symmetry, it suffices to show that $b\bar{f} = \bar{e}b\bar{f} \in A$.

Since $b \mathcal{L} f$ and $\mathcal{L}$ is a right congruence, we obtain $b\bar{f} \mathcal{L} f\bar{f} = \bar{f}$ so $b\bar{f} \in A$. Let g be an idempotent such that $g \mathcal{R} b\bar{f}$; then $b\bar{f} = g(b\bar{f})$ and $g = (b\bar{f})u$ for some $u \in S$. Consequently,

$$(ge)^2 = (b\bar{f}u)e(b\bar{f}u)e = b\bar{f}u(eb)\bar{f}ue = (b\bar{f}u)^2e = ge,$$

so $ge \in E_S$. Further, $g = b\bar{f}u = g(eb)\bar{f}u$ implies that $ge \mathcal{R} g$, so $ge \mathcal{R} b\bar{f}$. Since $b\bar{f} \in A$, it follows that $ge \in A$. Thus $ge \in E_A$ and $ge = b\bar{f}ue = (eb)\bar{f}ue = ege$ implies that $ge \leq e$. But then $ge \leq \bar{e}$ and we obtain

$$b\bar{f} = g(b\bar{f}) = g(eb)\bar{f} = (ge)b\bar{f} = \bar{e}(ge)b\bar{f} = \bar{e}b\bar{f}.$$

The second statement of the lemma follows from the first since there is no connection between e and f .

Because of this lemma, if $A \leq B$ we are able to define a function $\eta_{BA}: B \rightarrow A$ as follows:

$$\eta_{BA}: b \rightarrow \bar{e}b = b\bar{f} \quad \text{where } e \mathcal{R} b, b \mathcal{L} f \text{ } (b \in B).$$

Lemma 2.3. *If $A \leq B$, then η_{BA} satisfies (α) and (β) .*

Proof. Let $a \in A$, $b \in B$, and $e \in E_B$ be such that $ab \in A$, $e \mathcal{R} b$. Then $ab = a(eb) = (ae)b$ and $a(b\eta_{BA}) = a(\bar{e}b) = (a\bar{e})b$; thus to prove that $ab = a(b\phi)$, it suffices to show that $ae = a\bar{e}$. First note that $e \mathcal{R} b$ implies $ae \mathcal{R} ab$ so that $ae \in A$. Letting t be an inverse of ae , and $b = etae$, we obtain $b \in E_S$, $b \leq e$,

$b \mathcal{L} ae$ and hence $b \leq \bar{e}$. Consequently,

$$ae = ab = a(b\bar{e}) = (ab)\bar{e} = (ae)\bar{e} = a\bar{e}$$

as desired. Therefore $ab = a(b\eta_{BA})$; symmetrically $ba \in A$ implies $ba = (b\eta_{BA})a$. Since for any $e \in E_B$, we have $e\eta_{BA} = \bar{e}$, it follows that $E_B\eta_{BA} \subseteq E_A$. Q.E.D.

The next eight lemmas treat the relationships among the conditions $A \leq B$ and $A \preceq B$, the function η_{BA} , and functions ϕ susceptible to satisfying certain of the conditions (α)–(ε).

Lemma 2.4. *If $A \leq B$ and there exists ϕ satisfying (α) and (β), then $A \preceq B$.*

Proof. For any $e \in E_B$, we have $e\phi \in E_A$. If $f \in M_e \cap A$, then $f = fe = ef$ so that $f = (e\phi)f = f(e\phi)$ and thus $f \leq e\phi$. Hence $e\phi$ is the greatest element of $M_e \cap A$.

Lemma 2.5. *If $A \preceq B$ and A is also a $\mathcal{J}$ -class of S , then η_{BA} satisfies (γ).*

Proof. Since by 2.3 the function η_{BA} satisfies (α), by symmetry, it suffices to show that for $a \in A$, $b \in B$, the inclusion $a(b\eta_{BA}) \in A$ implies $ba \in A$. Indeed, for $f \in E_B$ such that $b \mathcal{L} f$, we obtain

$$J(a) = J(a(b\eta_{BA})) = J(ab\bar{f}) \subseteq J(ab) \subseteq J(a)$$

so that $ab \in A$.

Lemma 2.6. *If $A \preceq B$ and B^0 is completely 0-simple, then η_{BA} satisfies (δ).*

Proof. For $b, b', bb' \in B$, if $e \in E_B$ satisfies $e \mathcal{R} b$, then also $e \mathcal{R} bb'$ since B^0 is completely 0-simple. Hence

$$(bb')\eta_{BA} = \bar{e}(bb') = (\bar{e}b)b' = (b\eta_{BA})b'$$

and thus, by (α), also, $(b\eta_{BA})b' = (b\eta_{BA})(b'\eta_{BA})$ since $(bb')\eta_{BA} \in A$. Analogously, $(bb')\eta_{BA} = b(b'\eta_{BA})$. Q.E.D.

Lemmas 2.5 and 2.6 cease to be valid if both A and B are supposed to be only regular. For example, take again a simple regular ω -semigroup S which is not bisimple. In the notation of [5], letting $e_1 > e_2 > \dots > e_k$ be the poset of all idempotents of the chain of groups G , we specify $A = D_{(0, e_1, 0)}$, $B = D_{(0, e_2, 0)}$. A straightforward calculation shows that, for $a = (0, e_1, 0)$, $b = (0, e_2, 0)$, $b' = (1, e_2, 0)$, we have $a \in A$, $b, b' \in B$ and (1) $a(b\eta_{BA}) \in A$ but $ab \notin A$ so that (γ) fails, (2) $bb' \in B$ but $(b\eta_{BA})(b'\eta_{BA}) \neq (b\eta_{BA})b'$ so that (δ) fails.

Lemma 2.7. *If A^0 is completely 0-simple and ϕ is a function satisfying (α), then $A \preceq B$ and $\phi = \eta_{BA}$.*

Proof. If $e \in E_B$, $f, g \in E_A$ satisfy $e \geq f$, $e \geq g$, then, by (α), $f = (e\phi)f =$

$f(e\phi)$ which in a completely 0-simple semigroup implies that $e\phi = f$ and, analogously, $e\phi = g$. Consequently, $f = g$ and $A \preceq B$. Let $b \in B$ and $e, f \in E_B$ be such that $e \mathcal{R} b$, $b \mathcal{L} f$. Then $b\eta_{BA} = \bar{e}b \in A$ which by (α) implies $\bar{e}b = \bar{e}(b\phi)$ so that $b\eta_{BA} = \bar{e}(b\phi)$ and, symmetrically, $b\eta_{BA} = (b\phi)\bar{f}$. A simple computation using a Rees matrix representation of A^0 shows that the last two equations imply $b\phi = b\eta_{BA}$.

Lemma 2.8. *If $A \leq B$ and there exists a function ϕ satisfying (α) and (δ) , then $A \preceq B$ and $\phi = \eta_{BA}$.*

Proof. First note that (δ) implies (β) . Hence by 2.4 we have that $A \preceq B$ and by its proof that, for any $e \in E_B$, $e\phi$ is the greatest element of $M_e \cap E_A$. Consequently, $e\phi = \bar{e} = e\eta_{BA}$. For any $b \in B$, there is $e \in E_B$ such that $e \mathcal{R} b$, and hence by (δ) ,

$$b\phi = (eb)\phi = (e\phi)b = \bar{e}b = b\eta_{BA}.$$

Lemma 2.9. *If $A \preceq B$, $A \preceq C$, $b \in B$, $c \in C$, $bc \in A$, then η_{BA} and η_{CA} satisfy (ϵ) .*

Proof. For $e \in E_B$, $e \mathcal{R} b$, by (α) we obtain

$$bc = (eb)c = (e\eta_{BA})bc = \bar{e}(bc) = (\bar{e}b)c = (b\eta_{BA})c = (b\eta_{BA})(c\eta_{CA}),$$

and dually $bc = b(c\eta_{CA})$.

Lemma 2.10. *If $A \preceq B$, $B \preceq C$, $A \preceq C$, and A^0 and B^0 are completely 0-simple, then $\eta_{CB}\eta_{BA} = \eta_{CA}$.*

Proof. For $c \in C$, $e \in E_C$ such that $e \mathcal{R} c$, let e_B be the greatest element of $M_e \cap B$ and e_A be the greatest element of $M_{e_B} \cap A$. Using (δ) (see 2.6), we obtain

$$c\eta_{CB}\eta_{BA} = (e_Bc)\eta_{BA} = (e_B(e_Bc))\eta_{BA} = (e_B\eta_{BA})e_Bc = e_Ae_Bc = e_Ac.$$

On the other hand, $e \geq e_B \geq e_A$ and A^0 completely 0-simple imply that e_A is the only element in $M_e \cap A$, so that $c\eta_{CA} = e_Ac$ and the desired equality follows.

Lemma 2.11. *If $A = J_a$ and $B = J_b$, then $A \leq B$ if and only if $J(a) \subseteq J(b)$.*

Proof. If $A \leq B$, then for some $e \in E_A$, $f \in E_B$ we have $e \leq f$ so that $e \in J(f)$ and thus $J(a) = J(e) \subseteq J(f) = J(b)$. Conversely, suppose that $J(a) \subseteq J(b)$. For any $e \in E_A$ we have $e = xby$ for some $x, y \in S$. Then $e = exby$ and thus

$$J(e) = J(exby) \subseteq J(exb) \subseteq J(ex) \subseteq J(e)$$

which implies that $ex, (ex)b \in A$. Consequently, $AB \cap A \neq \emptyset$ and 2.1 implies

$A \leq B$.

As a consequence we have that if $\mathcal{D} = \mathcal{J}$ in a regular semigroup S then the relation $\leq$ agrees with the usual partial ordering of $\mathcal{J}$ -classes defined by $J_a \leq J_b$ if and only if $J(a) \subseteq J(b)$.

By T_A denote the trace of A and let χ_A be the homomorphism of S into the translational hull $\Omega(T_A)$ of T_A constructed in [14, §3]. In particular,

$$\ker \chi_A = (\ker \chi_A \pi_P) \cap (\ker \chi_A \pi_\Lambda)$$

where

$$\ker \chi_A \pi_P = \{(s, t) \mid \text{if } a \in A \text{ and either } as \mathcal{R} a \text{ or } at \mathcal{R} a, \text{ then } as = at\}$$

and for $\ker \chi_A \pi_\Lambda$ there is a dual expression.

Recall that an (ideal) extension V of a semigroup S is *dense* if the equality relation on V is the only congruence on V whose restriction to S is the equality on S (see, e.g., [3]). Also recall the notation $\pi_s = (\lambda_s, \rho_s)$.

Proposition 2.12. *If A is both a $\mathcal{D}$ - and $\mathcal{J}$ -class of a regular semigroup $S = S^0$, then $S\chi_A$ is a dense extension of the 0-bisimple regular semigroup $A\chi_A \cup \pi_0$.*

Proof. Let $F = A\chi_A \cup \pi_0$. If $a \in A$, $s \in S$, then

$$as \in J(a) = J_a \cup I(a) = A \cup I(a)$$

so that $(a\chi_A)(s\chi_A) = (as)\chi_A \in F$, and dually $(s\chi_A)(a\chi_A) \in F$, which proves that F is an ideal of $S\chi_A$. Since A is a regular $\mathcal{D}$ -class and $A\chi_A \cup \pi_0 = F$ is an ideal of $S\chi_A$, it follows that $A\chi_A$ is a regular $\mathcal{D}$ -class of $S\chi_A$, so F is a 0-bisimple regular semigroup. To prove that the extension is dense, we suppose that, for $s, t \in S$,

$$(s\chi_A)(a\chi_A) = (t\chi_A)(a\chi_A), \quad (a\chi_A)(s\chi_A) = (a\chi_A)(t\chi_A) \quad (a \in A).$$

Then $(sa, ta), (as, at) \in \ker \chi_A$ ($a \in A$). Let $a \in A$ and suppose $as \mathcal{R} a$. For $e \in E_S$ such that $e \mathcal{R} a$, we get $ea = a$ and $e \mathcal{R} as$ so that $e \mathcal{R} e(as)$. Since $(as, at) \in \ker(\chi_A \pi_P)$, it follows that $eas = eat$ and hence $as = at$. Symmetrically, $at \mathcal{R} a$ implies $as = at$. Dually, either of the relations $sa \mathcal{L} a$, $ta \mathcal{L} a$ implies $sa = ta$. But then $(s, t) \in \ker \chi_A$ which proves that $s\chi_A = t\chi_A$, which by [3, 3.7] implies that $S\chi_A$ is a dense extension of F .

The conclusion of 2.12 fails for all but one $\mathcal{D}$ -class of a simple but not bi-simple regular ω -semigroup S (again by a straightforward computation using the representation in [5] or [10]).

Corollary 2.13. *Every regular semigroup $S = S^0$ in which $\mathcal{D} = \mathcal{J}$ is a subdirect product of dense extensions of 0-bisimple regular semigroups.*

Proof. This follows from [14, Proposition 2] and 2.12.

We are now interested in the case $S\chi_A = A\chi_A \cup \pi_0$ for a regular semigroup S , for if this holds for all $\mathcal{D}$ -classes, then S is a subdirect product of 0-bisimple regular semigroups $A\chi_A \cup \pi_0$.

Lemma 2.14. $B\chi_A \subseteq A\chi_A$ if and only if there exists a function ϕ satisfying (y).

Proof. Necessity. The hypothesis implies that for every $b \in B$ there exists $a \in A$ such that $b\chi_A = a\chi_A$. Since $\chi_A|_A$ is one-to-one, a is unique and we write $\phi: b \rightarrow a = \bar{b}$. Hence $\lambda^b = \lambda^{\bar{b}}$ and thus for every $(a; i, \mu) \in T_A$, we have

$$(1) \quad bH_{i1} = H_{j1} \Leftrightarrow \bar{b}H_{i1} = H_{j1} \Rightarrow r'_j br_i = r'_j \bar{b}r_i.$$

Let $a \in A$, then $a \in H_{i\mu}$ for some i and μ , and thus $a = r_i u$ for some $u \in S$. Suppose next that $ba \in A$. Then $ba = br_i u \in A$ and since $br_i u \mathcal{R} br_i$, it follows that $br_i \in A$. Hence $bH_{i1} = H_{j1}$ for some j and (1) implies that $\bar{b}H_{i1} = H_{j1}$ and $r'_j br_i = r'_j \bar{b}r_i$. By virtue of $r_j r'_j = e_j \mathcal{R} br_i$ and $e_j \mathcal{R} \bar{b}r_i$, we obtain

$$ba = b(r_i u) = e_j (br_i) u = r_j (r'_j br_i) u = r_j (r'_j \bar{b}r_i) u = e_j (\bar{b}r_i) u = \bar{b}(r_i u) = \bar{b}a.$$

Conversely, if $\bar{b}a \in A$, an analogous proof shows that then $\bar{b}a = ba$. This establishes the second part of (y) for the function ϕ ; the first part follows similarly from $\rho^b = \rho^{\bar{b}}$.

Sufficiency. For any $b \in B$, again writing $\bar{b} = b\phi$, if $bH_{i1} = H_{j1}$, then $\bar{b}r_i = br_i \in H_{j1}$ and hence $r'_j br_i = r'_j \bar{b}r_i \in H_{11}$. Similarly, if $\bar{b}H_{i1} = H_{j1}$, then $r'_j br_i = r'_j \bar{b}r_i \in H_{11}$. Consequently, $\lambda^b = \lambda^{\bar{b}}$, and dually $\rho^b = \rho^{\bar{b}}$ and thus $b\chi_A = \bar{b}\chi_A$.

The following concept will prove very useful.

Definition 2.15. Let C be a nonempty subset of a semigroup S . Then S satisfies C -majorization if, for any $e \in E_S$, either $M_e \cap C = \emptyset$ or $M_e \cap C$ has a greatest element.

Theorem 2.16. Let $S = S^0$ be a regular semigroup and let A be a $\mathcal{D}$ -class of S . If $S\chi_A = A\chi_A \cup \pi_0$, then S satisfies A -majorization; the converse holds if A is also a $\mathcal{J}$ -class of S .

Proof. Suppose that $S\chi_A = A\chi_A \cup \pi_0$ and let B be an arbitrary $\mathcal{D}$ -class of S . If $A \leq B$ does not hold, then, for every $e \in E_B$, we have $M_e \cap A = \emptyset$. If $A \leq B$, then $B\chi_A \subseteq A\chi_A$ since χ_A maps B into a $\mathcal{D}$ -class of $A\chi_A \cup \pi_0$, and 2.14 implies the existence of a function $\phi: B \rightarrow A$ satisfying (y). Since (y) implies both (α) and (β), by 2.4 we have that $A \leq B$.

Now suppose that A is also a $\mathcal{J}$ -class of S and that S satisfies A -majorization. Again let B be any $\mathcal{D}$ -class of S . If $A \leq B$ does not hold, then $B\chi_A = \pi_0$. If $A \leq B$, then 2.5 implies that η_{BA} satisfies (y) which by 2.14 yields $B\chi_A \subseteq A\chi_A$. Consequently, $S\chi_A = A\chi_A \cup \pi_0$.

We now introduce the notion which for the case $\sigma = \mathcal{D}$ will be a fundamental hypothesis for a great number of statements throughout the paper.

Definition 2.17. Let σ be an equivalence relation on a semigroup S . Then S satisfies σ -majorization if S satisfies C -majorization for all equivalence classes C of σ .

Corollary 2.18. Every regular semigroup $S = S^0$ satisfying $\mathcal{D} = \mathcal{J}$ and $\mathcal{D}$ -majorization is a subdirect product of 0-bisimple regular semigroups.

Proof. Let Δ be an index set of $\mathcal{D}$ -classes of S different from 0. Then [14, Proposition 2] says that the mapping $\chi: s \rightarrow (s\chi_D)_{D \in \Delta}$ is an isomorphism of S into the direct product $\prod_{D \in \Delta} \Omega(T_D)$ (we have omitted here the $\mathcal{D}$ -class 0 as we may). Consequently, S is a subdirect product of semigroups $S\chi_A$, $A \in \Delta$. Since for every $A \in \Delta$, by hypothesis A is also a $\mathcal{J}$ -class and S satisfies A -majorization, 2.16 implies that $S\chi_A = A\chi_A \cup \pi_0$ so that $S\chi_A$ is a 0-bisimple regular semigroup.

Theorem 2.16 and its corollary have their obvious analogues for the case of a regular semigroup without zero. Indeed, if A in 2.16 is the kernel of S , then $S\chi_A = A\chi_A$ should be substituted for $S\chi_A = A\chi_A \cup \pi_0$, otherwise the statement remains the same. Corollary 2.18 for the general case becomes

Corollary 2.19. Every regular semigroup satisfying $\mathcal{D} = \mathcal{J}$ and $\mathcal{D}$ -majorization is a subdirect product of 0-bisimple and bisimple regular semigroups.

3. Subdirect products of completely 0-simple semigroups. Recall that for any semigroup S , the inner part $\Pi(S)$ of the translational hull $\Omega(S)$ of S is the ideal of $\Omega(S)$ consisting of all $\pi_s = (\lambda_s, \rho_s)$, $s \in S$, where $\lambda_s x = sx$, $x\rho_s = xs$ for all $x \in S$ (see, e.g., [3]). We are interested here in conditions on a regular $\mathcal{D}$ -class D of a semigroup S in order that $S\chi_D = \Pi(T_D)$. As a consequence we deduce a slight improvement of an important theorem due to Lallement [6], giving a simpler proof.

Lemma 3.1. Let D be a regular $\mathcal{D}$ -class of a semigroup $S = S^0$. Then $D\chi_D = \Pi(T_D) \setminus \pi_0$ if and only if D^0 is a completely 0-simple semigroup.

Proof. Necessity. The function $\chi_D|_D$ is a partial isomorphism of D onto $\Pi(T_D) \setminus \pi_0$ since χ_D is a homomorphism of S whose restriction to D is one-to-one. For $e, f \in E_D$, $e \leq f$ implies $e\chi_D \leq f\chi_D$ and, since $\Pi(T_D)$ is completely 0-simple, we must have $e\chi_D = f\chi_D$. But then $e = f$, proving that every nonzero idempotent of D^0 is primitive. Since D^0 is 0-bisimple, the Rees theorem implies that D^0 is completely 0-simple.

Sufficiency. For $s \in D$ with $s \in R_j \cap L_\nu$, let $\bar{s} = (r'_j s q'_\nu; j, \nu)$; then

$$\begin{aligned}
 (1) \quad \lambda^s(a; i, \mu) &= (r'_k s r'_i a; k, \mu) \quad \text{if } sH_{i1} = H_{k1} \\
 &= 0 \quad \text{otherwise,} \\
 \lambda_{\overline{s}}(a; i, \mu) &= (r'_j s q'_\nu; j, \nu)(a; i, \mu) = (r'_j s q'_\nu(q_\nu r'_i) a; j, \mu) \\
 &= (r'_j s(q'_\nu q_\nu) r'_i a; j, \mu) \\
 (2) \quad &= (r'_j s r'_i a; j, \mu) \quad \text{if } q_\nu r'_i \in H_{11}, \\
 &= 0 \quad \text{otherwise.}
 \end{aligned}$$

Now giving D^0 the same Rees matrix representation used in constructing T_D , we have $s = (g; j, \nu)$, $q_\nu = (b; 1, \nu)$, $r'_i = (t; i, 1)$ for some $g, b, t \in H_{11}$, and obtain

$$\begin{aligned}
 s r'_i &= (g; j, \nu)(t; i, 1) = (g p_{\nu i} t; j, 1), \\
 q_\nu r'_i &= (b; 1, \nu)(t; i, 1) = (b p_{\nu i} t; 1, 1),
 \end{aligned}$$

which implies $sH_{i1} = H_{k1} \Leftrightarrow q_\nu r'_i \in H_{11} \Rightarrow k = j$. This together with (1) and (2) shows that $\lambda^s(a; i, \mu) = \lambda_{\overline{s}}(a; i, \mu)$. Consequently, $\lambda^s = \lambda_{\overline{s}}$ and a symmetric proof shows that also $\rho^s = \rho_{\overline{s}}$, so that $s\chi_D = \pi_{\overline{s}}$. It is easy to see that for $(g; j, \nu) \in T_D$, letting $s = r_j g q_\nu$, we obtain $s\chi_D = \pi_{\overline{s}} = \pi_{(g;j,\nu)}$. Therefore, χ_D maps D onto $\Pi(T_D) \setminus \pi_0$.

Note that in the proof of necessity it is enough to assume that $D\chi_D \subseteq \Pi(T_D) \setminus \pi_0$.

Lemma 3.2. *If D is a $\mathcal{D}$ -class of a semigroup S and D^0 is completely 0-simple, then D is a $\mathcal{J}$ -class of S .*

Proof. Let J be the $\mathcal{J}$ -class of S containing D . Then the principal factor J^0 is 0-simple, and if it is not completely 0-simple, by [1, Theorem 2.54], for every idempotent e in D , J^0 contains a copy B of the bicyclic semigroup with identity e . But then for some $f \in E_B$, we have $e > f$ and $e \mathcal{D} f$, implying that D contains two comparable nonzero idempotents. This contradicts the hypothesis that D^0 is completely 0-simple. Hence J^0 is completely 0-simple and $D = J$.

Theorem 3.3. *Let D be a $\mathcal{D}$ -class of a regular semigroup $S = S^0$ with $D \neq 0$. Then $S\chi_D = \Pi(T_D)$ if and only if D^0 is completely 0-simple and S satisfies D -majorization.*

Proof. Necessity. If for some $s \in D$ we have $s\chi_D = \pi_0$, then since χ_D is a homomorphism, we also have $D\chi_D = \pi_0$. If D has only one element, say x , then x is an idempotent and $x\chi_D = \pi_0$ is impossible, and if D has more than one element, $D\chi_D = \pi_0$ contradicts the fact that χ_D restricted to D is one-to-one. Hence $D\chi_D \subseteq \Pi(T_D) \setminus \pi_0$ and 3.1, together with the remark following it, implies that D^0 is completely 0-simple, which in turn, again by 3.1, implies that $D\chi_D =$

$\Pi(T_D) \setminus \pi_0$. It follows that $S\chi_D = \Pi(T_D) = D\chi_D \cup \pi_0$ which by 2.16 implies that S satisfies D -majorization.

Sufficiency. Since D^0 is completely 0-simple, 3.1 implies that $D\chi_D \cup \pi_0 = \Pi(T_D)$ and 3.2 that D is also a $\mathcal{J}$ -class of S . Hence by 2.16, we have $S\chi_D = D\chi_D \cup \pi_0 = \Pi(T_D)$.

Recall that a semigroup S is said to be *completely semisimple* if each principal factor of S is completely (0-)simple.

Theorem 3.4 (cf. [6, 2.12, 2.17]). *The following conditions on a regular semigroup S are equivalent.*

(a) S is completely semisimple and satisfies $\mathcal{D}$ -majorization.

(b) S is completely semisimple and, for any elements $a, b \in S$, $J(a) \subseteq J(b)$ implies $D_a \leq D_b$ and the existence of a function $\phi: D_b \rightarrow D_a$ satisfying (a).

(c) S is a subdirect product of completely (0-)simple semigroups.

The function ϕ in (b) is equal to $\eta_{D_b D_a}$ and satisfies (β), (γ), (δ), and the system of functions $\{\eta_{D_b D_a}\}$ satisfies (ε) and has the transitivity property in 2.10.

Proof. (a) $\Rightarrow$ (b). This follows from 2.11 and 2.3.

(b) $\Rightarrow$ (a). This follows from 2.11 and 2.7.

(a) $\Rightarrow$ (c). It follows from 3.3 that S is a subdirect product of completely (0-)simple semigroups $\Pi(T_D)$, where D ranges over all $\mathcal{D}$ -classes of S .

(c) $\Rightarrow$ (a). Assume that S is a subdirect product of completely (0-)simple semigroups S_α , $\alpha \in A$. Then S is completely semisimple by [6, Lemma 2.14]. We suppose (as we may) that $S \subseteq \prod_{\alpha \in A} S_\alpha$ and also that $e, f, g \in E_S$ have the properties $e \geq f$, $e \geq g$, $f \mathcal{D} g$. Writing $e = (e_\alpha)$, $f = (f_\alpha)$, $g = (g_\alpha)$, it follows that for every $\alpha \in A$, $e_\alpha \geq f_\alpha$, $e_\alpha \geq g_\alpha$, $f_\alpha = 0 \Leftrightarrow g_\alpha = 0$. Hence if $f_\alpha \neq 0$, then $f_\alpha \leq e_\alpha$ implies $f_\alpha = e_\alpha$, and $g_\alpha \neq 0$ so $g_\alpha \leq e_\alpha$ implies $g_\alpha = e_\alpha$, hence $f_\alpha = g_\alpha$. Consequently, $f = (f_\alpha) = (g_\alpha) = g$ which establishes $\mathcal{D}$ -majorization.

When (b) holds, for $A = D_a$ and $B = D_b$, by 2.11 and 2.7 we obtain $A \leq B$ and $\phi = \eta_{BA}$. Hence ϕ satisfies (γ) by 2.5 and (δ) by 2.6 ((δ) trivially implies (β)), and the system $\{\eta_{BA}\}$ has the required properties by 2.9 and 2.10.

4. Completely regular semigroups. We will now apply some of the results in the preceding section to this class of semigroups, obtain additional characterizations and a construction of the semigroups in question. The results here supplement some of those of Lallement [6, Chapitre II, §4]. Recall that a semigroup S is called *completely regular* if for every $a \in S$ there exists $b \in S$ such that $a = aba$, $ab = ba$; equivalently, S is a union of (pairwise disjoint) groups (see [1], [9]). A band (idempotent semigroup) B is called (*left*, resp. *right*) *normal* if it satisfies the identity ($axy = ayx$, resp. $xya = yxa$) $axy = ayx$ (see, e.g., [17]). Note that in a completely regular semigroup S , the relation $\mathcal{D}$ is the smallest semilattice

congruence on S and each $\mathcal{D}$ -class is completely simple, so S is a semilattice of completely simple semigroups (and conversely). In accordance with this terminology, we say that a semigroup S is a *normal band of groups* if $\mathcal{H}$ is a congruence, $S/\mathcal{H}$ is a normal band, and every $\mathcal{H}$ -class of S is a group. The last requirement is equivalent to the fact that S is completely regular. If a is an element of a completely regular semigroup S , we denote by a^* the identity of the maximal subgroup of S containing a .

Theorem 4.1 (cf. [6, 2.22, 2.23]). *The following conditions on a completely regular semigroup S are equivalent.*

- (a) S satisfies $\mathcal{D}$ -majorization.
- (b) Both $\mathcal{L}$ and $\mathcal{R}$ are congruences, $S/\mathcal{L}$ is a right normal band, $S/\mathcal{R}$ is a left normal band.
- (c) S is a normal band of groups.
- (d) For any $e, f, g \in E_S$, $e \leq f$ implies $(eg)^* \leq (fg)^*$, $(ge)^* \leq (gf)^*$.
- (e) S is a subdirect product of completely simple semigroups with a zero possibly adjoined.

Proof. (a) $\Rightarrow$ (b). Since $\mathcal{R}$ is always a left congruence, it suffices to show that it is here a right congruence. Thus let $a, b, c \in S$, $a \mathcal{R} b$, b' be an inverse of b , $f = bb'$, t be an inverse of bc , D be the $\mathcal{D}$ -class of S containing bc . Then $bctbb' \in E_D$ and $f \geq bctbb'$. Also $ac, cb' \in D$ and thus there exists $v \in D$ such that $acvcb' \in E_D$ since D is completely simple. Now $a = bx$ for some $x \in S$ and thus $fa = bb'a = bb'(bx) = bx = a$ so that $f \geq acvcb'$. Consequently, the hypothesis implies that $bctbb' = acvcb'$ and hence

$$bc = (bc)t(bc) = (bctbb')(bc) = (acvcb')(bc) \in acS$$

and, dually, $ac \in bcS$. Therefore, $ac \mathcal{R} bc$ which proves that $\mathcal{R}$ is a congruence.

Now let $a, x, y \in S$, let a' be an inverse of a , $e = aa'$, s be an inverse of axy , D be the $\mathcal{D}$ -class of S containing axy . Then $axysaa' \in E_D$, $ayx, xya' \in D$ and there exists $u \in D$ such that $ayxuxya' \in E_D$ since D is completely simple. Further, $e \geq axysaa'$ and $e \geq ayxuxya'$ which by the hypothesis yields $axysaa' = ayxuxya'$. Consequently,

$$axy = axysaxy = (axysaa')(axy) = (ayxuxya')(axy) \in ayxS$$

and, dually, $ayx \in axyS$. Hence $axy \mathcal{R} ayx$ which shows that $S/\mathcal{R}$ is a left normal band.

The statements concerning $\mathcal{L}$ are established in an analogous manner.

(b) $\Rightarrow$ (c). Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, we have that $\mathcal{H}$ is a congruence and that $S/\mathcal{H}$ is a subdirect product of the right normal band $S/\mathcal{L}$ and the left normal band $S/\mathcal{R}$. Hence we may consider $S/\mathcal{H}$ as a subsemigroup of $(S/\mathcal{L}) \times (S/\mathcal{R})$ and obtain

$$\begin{aligned}(a_1, a_2)(x_1, x_2)(y_1, y_2)(a_1, a_2) &= (a_1 x_1 y_1 a_1, a_2 x_2 y_2 a_2) \\ &= (a_1 y_1 x_1 a_1, a_2 y_2 x_2 a_2) = (a_1, a_2)(y_1, y_2)(x_1, x_2)(a_1, a_2).\end{aligned}$$

Thus $S/\mathcal{H}$ is a normal band.

(c) $\Rightarrow$ (d). First let B be a normal band, let $a, b, c \in B$ with $a \leq b$. Then $ac = (ab)c = a(bc)c = (ac)(bc)$ and similarly $ac = (bc)(ac)$ so that $ac \leq bc$. Next let $s \rightarrow s'$ be the canonical homomorphism of S onto $B = S/\mathcal{H}$, let $e, f, g \in E_S$, and suppose that $e \leq f$. Then $e' \leq f'$ and hence $e'g' \leq f'g'$ so that $(eg)' \leq (fg)'$. Consequently, $(eg)' = (eg)'(fg)' = (fg)'(eg)'$ and thus

$$(1) \quad (eg)^* \mathcal{H} (eg)^*(fg)^*, \quad (eg)^* \mathcal{H} (fg)^*(eg)^*.$$

But then

$$(eg)^*(fg)^* = [(eg)^*(fg)^*](eg)^* = (eg)^*[(fg)^*(eg)^*] = (fg)^*(eg)^*$$

and hence both $(eg)^*(fg)^*$ and $(fg)^*(eg)^*$ are idempotents. It then follows from (1) that

$$(eg)^* = (eg)^*(fg)^* = (fg)^*(eg)^*,$$

i.e. $(eg)^* \leq (fg)^*$. The proof of $(ge)^* \leq (gf)^*$ is dual.

(d) $\Rightarrow$ (a). Let $e, f, g \in E_S$ with $e \geq f$, $e \geq g$, $f \mathcal{D} g$. Then $g = ge = (ge)^* \geq (gf)^*$ and similarly $g \geq (fg)^*$. Since g, gf, fg are all $\mathcal{D}$ -related, it follows that $g = (gf)^* = (fg)^*$. Analogously $f = (gf)^* = (fg)^*$ which yields $f = g$.

(a) $\Rightarrow$ (e). We have seen in the preceding section that such a semigroup S is a subdirect product of semigroups $\Pi(T_D)$ as D ranges over all $\mathcal{D}$ -classes of S . Since for no $\mathcal{D}$ -class D , the semigroup D^0 has zero divisors, the same holds for the semigroups $\Pi(T_D)$.

(e) $\Rightarrow$ (a). This follows from 3.4.

Using 3.4 it is easy to see that the semigroups in 4.1, and only they, can be constructed as follows (cf. [17, Theorem 6]; [6, 2.21, 2.22]).

Construction 4.2. Let Y be a semilattice, for each $\alpha \in Y$, let S_α be a completely simple semigroup, and suppose that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For $\alpha, \beta \in Y$ with $\alpha > \beta$, let $\theta_{\alpha, \beta}: S_\alpha \rightarrow S_\beta$ be a homomorphism, let $\theta_{\alpha, \alpha}$ be the identity mapping on S_α , and suppose that

$$(2) \quad \theta_{\alpha, \beta} \theta_{\beta, \gamma} = \theta_{\alpha, \gamma} \quad \text{if } \alpha > \beta > \gamma.$$

On the set $S = \bigcup_{\alpha \in Y} S_\alpha$ define a multiplication by

$$a * b = (a \theta_{\alpha, \alpha\beta})(b \theta_{\beta, \alpha\beta}) \quad (a \in S_\alpha, b \in S_\beta).$$

Then S satisfies the conditions of 4.1, and conversely, every such semigroup can be obtained in this way.

For a given semilattice Y and sets S_α with functions $\theta_{\alpha,\beta}$ as above, call the system $\{\theta_{\alpha,\beta}\}$ *transitive* if it satisfies (2).

Corollary 4.3 (cf. [4]). *The following conditions on a completely regular semigroup S are equivalent.*

- (a) S satisfies the conditions 4.1 and E_S is a subsemigroup of S .
- (b) E_S is a normal band.
- (c) S is a subdirect product of a class of semigroups of the following types: groups and left or right zero semigroups, and any of these with a zero adjoined.

Proof. (a) $\Rightarrow$ (c). By 4.1, S is a subdirect product of semigroups T and/or T^0 with T completely simple, where now E_T forms a subsemigroup of T since T or T^0 is a homomorphic image of S . But then T is isomorphic to the direct product $G \times L \times R$ where G, L, R are a group, and a left and a right zero semigroup, respectively (for a proof, see, e.g., [11, Lemme 1]). Hence $T^0 \cong (G \times L \times R)^0$ where the latter is a subdirect product of semigroups G^0, L^0, R^0 .

(c) $\Rightarrow$ (b). Since in each of the semigroups listed in (c) idempotents form a normal band, the same holds for any of their subdirect products.

(b) $\Rightarrow$ (a). By 4.1, E_S satisfies $\mathcal{D}$ -majorization, which evidently implies that the same holds for S , and thus S satisfies the conditions of 4.1.

It follows from (c) that subdirectly irreducible groups, 2-element left or right zero semigroups, and all of these with a zero adjoined constitute the class of all subdirectly irreducible semigroups satisfying the conditions of 4.3.

In order to give a construction of the semigroups satisfying the conditions of 4.3, in the above construction we take $S_\alpha = G_\alpha \times L_\alpha \times R_\alpha$ and compute all homomorphisms of such a semigroup into another of the same type. This can be done by specializing the expression of a homomorphism of an arbitrary completely (0-) simple semigroup into another (see, e.g., [1, Theorem 3.11]) or by a direct consideration as in [12]. All homomorphisms $\theta: G \times L \times R \rightarrow G' \times L' \times R'$ are given as follows: let $\omega: G \rightarrow G'$ be a homomorphism, $\phi: L \rightarrow L'$ and $\psi: R \rightarrow R'$ be functions, and let $(g, l, r)\theta = (g\omega, l\phi, r\psi)$.

Construction 4.4. Let Y be a semilattice, for each $\alpha \in Y$, let G_α be a group, L_α and R_α be nonempty sets, and suppose that $G_\alpha \cap G_\beta = L_\alpha \cap L_\beta = R_\alpha \cap R_\beta = \emptyset$ if $\alpha \neq \beta$. For $\alpha, \beta \in Y$ with $\alpha > \beta$, let $\omega_{\alpha,\beta}: G_\alpha \rightarrow G_\beta$ be a homomorphism, $\phi_{\alpha,\beta}: L_\alpha \rightarrow L_\beta$ and $\psi_{\alpha,\beta}: R_\alpha \rightarrow R_\beta$ be functions, let $\omega_{\alpha,\alpha}, \phi_{\alpha,\alpha}, \psi_{\alpha,\alpha}$ be the identity functions on $G_\alpha, L_\alpha, R_\alpha$, respectively, and suppose that each of the systems $\{\omega_{\alpha,\beta}\}, \{\phi_{\alpha,\beta}\}, \{\psi_{\alpha,\beta}\}$ is transitive. On the set $S = \bigcup_{\alpha \in Y} (G_\alpha \times L_\alpha \times R_\alpha)$ define a multiplication as follows: for $(g, l, r) \in G_\alpha \times L_\alpha \times R_\alpha$ and $(g', l', r') \in G_\beta \times L_\beta \times R_\beta$,

$$(g, l, r) * (g', l', r') = ((g\omega_{\alpha, \alpha\beta})(g'\omega_{\beta, \alpha\beta}), l\phi_{\alpha, \alpha\beta}, r'\psi_{\beta, \alpha\beta}).$$

Then S satisfies the conditions of 4.3, and conversely, every such semigroup can be obtained in this way.

This subject is related to the theme of the structure of regular semigroups satisfying permutational identities, see [16].

5. Conditions on ideals of a regular semigroup. We now return to regular semigroups and ask the following question: what can we say about ideals of a regular semigroup S if S satisfies $\mathcal{D}$ -majorization or E_S is a tree? The theorems below give an answer to this question for S suitably (and, indeed, strongly) restricted; the converse of these results is not true, but a partial converse will be established in §7. To this end, we first introduce a new concept.

Definition 5.1. Let I be an ideal of S and σ be a binary relation on a semigroup S . Then I is a σ -categorical ideal of S if, for any $a, b, c \in S$, $ab, bc \notin I$ and $ab \sigma bc$ imply $abc \notin I$.

When σ is the universal relation, a σ -categorical ideal is the usual *categorical ideal*; if S has a zero which is a categorical ideal of S , then S is said to be *categorical at zero* (see [1, §7.7], where S itself is not a categorical ideal, we do not make this exception).

Lemma 5.2. *The following conditions on an arbitrary semigroup S are equivalent.*

- (a) For all $a, b, c \in S$, if $ab \mathcal{J} bc$ then $abc \mathcal{J} ab$ [either $abc \mathcal{J} ab$ or $abc \mathcal{J} bc$].
- (b) All principal ideals of S are $\mathcal{J}$ -categorical [categorical].
- (c) All ideals of S are $\mathcal{J}$ -categorical [categorical].

Proof. We will prove only the unbracketed statements, the bracketed statements are treated analogously.

(a) $\Rightarrow$ (b). If $ab, bc \notin J(x)$ and $ab \mathcal{J} bc$, then $abc \mathcal{J} ab$ and thus $abc \notin J(x)$.

(b) $\Rightarrow$ (c). Let I be an ideal of S and $ab, bc \notin I$, $ab \mathcal{J} bc$. For every $x \in I$, the principal ideal $J(x)$ is $\mathcal{J}$ -categorical and $J(x) \subseteq I$. Thus $ab, bc \notin J(x)$, and hence $ab \mathcal{J} bc$ implies $abc \notin J(x)$. Since $x \in I$ is arbitrary, it follows that $abc \notin I$ and I is $\mathcal{J}$ -categorical.

(c) $\Rightarrow$ (a). If $ab \mathcal{J} bc$, then $ab, bc \notin I(ab)$ and since $I(ab)$ is either $\mathcal{J}$ -categorical or empty, we must have $abc \notin I(ab)$ so that $abc \mathcal{J} ab$.

Theorem 5.3. *If a regular semigroup $S = S^0$ satisfies (a) $\mathcal{D} = \mathcal{J}$, (b) all principal factors of S are categorical at zero, (c) $\mathcal{D}$ -majorization, then all ideals of S are $\mathcal{D}$ -categorical.*

Proof. For any $\mathcal{D}$ -classes A and B , if $A \leq B$, then the hypothesis implies

$A \leq B$, so by 2.3 the function $\eta_{BA} : B \rightarrow A$ satisfies (α) and (β) , and by 2.9 the system of functions $\{\eta_{BA}\}$ satisfies (ϵ) . Now suppose that $ab \mathcal{D} bc$. Letting $\alpha = \eta_{D_a D_{ab}}$, $\beta = \eta_{D_b D_{ab}}$, $\gamma = \eta_{D_c D_{ab}}$, we obtain $ab = (a\alpha)(b\beta)$, $bc = (b\beta)(c\gamma)$. It follows that $a\alpha, b\beta, c\gamma, (a\alpha)(b\beta), (b\beta)(c\gamma) \in D_{ab}$ where D_{ab} can be considered as the set of all nonzero elements of the principal factor $J(ab)/I(ab)$. By hypothesis, the latter is categorical at zero which implies that $(a\alpha)(b\beta)(c\gamma) \in D_{ab}$. By (ϵ) we also have $bc = (b\beta)c$ and thus $abc = (a\alpha)(b\beta)c = (a\alpha)(b\beta)(c\gamma) \in D_{ab}$. Hence $abc \mathcal{D} ab$ and 5.2 implies that all ideals of S are $\mathcal{D} = \mathcal{J}$ -categorical.

Recall that a partially ordered set P is called a *tree* if, for any $a, b, c \in P$, the relations $a \geq b$, $a \geq c$ imply that either $b \geq c$ or $c \geq b$.

Theorem 5.4. *If S is a completely semisimple semigroup for which E_S is a tree, then all ideals of S are categorical.*

Proof. By 5.2 it suffices to show that for any $a, b, c \in S$, we have $abc \in D_{ab} \cup D_{bc}$. Since $J(ab) \subseteq J(b)$ and $J(bc) \subseteq J(b)$, by 2.11 there exist idempotents $e \in D_b$, $f \in D_{ab}$, $g \in D_{bc}$ such that $e \geq f$, $e \geq g$. The hypothesis implies that either $f \geq g$ or $g \geq f$; suppose that $f \geq g$, the other case is treated analogously. Then $J(bc) \subseteq J(ab) \subseteq J(b)$ which by 2.11 and the hypothesis implies $D_{bc} \leq D_{ab}$, $D_{ab} \leq D_b$, $D_{bc} \leq D_b$. Now let $\phi = \eta_{D_b D_{ab}}$, $\psi = \eta_{D_{ab} D_{bc}}$, $\chi = \eta_{D_b D_{bc}}$, and let b' be an inverse of b . Using 2.3, 2.6, and 2.10, we obtain

$$(1) \quad (ab)\psi = (abb'b)\psi = [(abb')(b\phi)]\psi = (abb')\psi(b\phi\psi) = (abb')\psi(b\chi),$$

$$bc = b(b'bc) = (b\chi)(b'bc),$$

so that $(abb')\psi, b\chi, b'bc, (abb')\psi(b\chi), (b\chi)(b'bc) \in D_{bc}$. Since D_{bc}^0 is completely 0-simple and thus categorical at zero, it follows that $(abb')\psi(b\chi)(b'bc) \in D_{bc}$ and (1) implies $(ab)\psi(b'bc) \in D_{bc}$. Here $ab \in D_{ab}$, $b'bc \in D_{bc}$, so by 2.5 we conclude that

$$abc = (ab)(b'bc) = (ab)\psi(b'bc) \in D_{bc}$$

as desired.

If S is an extension of a completely simple semigroup by a completely 0-simple semigroup, then all ideals of S^0 are categorical but we cannot say anything about the idempotents of S^0 . Thus the converses of both 5.3 and 5.4 fail badly.

6. Extensions of a primitive regular semigroup without contractions. In order to establish a converse of the theorems in the preceding section, we consider here a more general situation, namely that of extensions determined by partial homomorphisms of a primitive regular semigroup without contractions. For the pertinent material consult [1, Chapter 3].

Definition 6.1. A regular $\Lambda \times I$ matrix P over a group with zero has no contractions if

$$p_{\lambda i} \neq 0 \Leftrightarrow p_{\mu i} \neq 0 \text{ for every } i \in I \text{ implies } \lambda = \mu,$$

$$p_{\lambda i} \neq 0 \Leftrightarrow p_{\lambda j} \neq 0 \text{ for every } \lambda \in \Lambda \text{ implies } i = j.$$

Thus P has no contractions if and only if no two distinct rows and no two distinct columns of P have the corresponding entries simultaneously different from zero. The case of P without contractions may be considered as a weakening of the condition that P be an invertible matrix and also as the opposite extreme of the case in which all entries of P are different from zero.

Recall that a completely 0-simple semigroup T with trivial subgroups is called a *rectangular 0-band*, and that the congruence induced on an arbitrary semigroup S by a homomorphism ϕ of S onto T is called an *l -matrix congruence* where $l = 0\phi^{-1}$ (see [7]). A congruence on a semigroup different from the universal relation is called *proper*. The next proposition will be quite useful.

Proposition 6.2. *The following conditions on a regular Rees matrix semigroup $S = \mathfrak{M}^0(G; I, \Lambda; P)$ are equivalent.*

- (a) P has no contractions.
- (b) $\mathcal{H}$ is the only 0-matrix congruence on S .
- (c) Every proper congruence on S is contained in $\mathcal{H}$.
- (d) For $e, f \in E_S$, $exe \neq 0 \Leftrightarrow fxf \neq 0$ for all $x \in S$ implies $e = f$.

Proof. (a) $\Rightarrow$ (b). By [6, 4.24] every proper congruence on S is obtained from a triple (N, p, π) where N is a normal subgroup of G , p and π are partitions of I and Λ , respectively. The conditions in [6, 4.22] on p and π in conjunction with our hypothesis imply that both must be the equality relations. On the other hand, the congruence associated with (N, p, π) is a 0-matrix congruence if and only if $N = G$. Consequently the only 0-matrix congruence on S is the one associated with (G, i_I, ι_Λ) where i_I and ι_Λ are the equality relations on I and Λ , respectively, namely $\mathcal{H}$.

(b) $\Rightarrow$ (c). By [6, 4.22, 4.24] and the hypothesis, only the triples of the form (N, i_I, ι_Λ) are associated with proper congruences on S , since otherwise (G, p, π) for either $p \neq i_I$ or $\pi \neq \iota_\Lambda$ would yield a 0-matrix congruence different from $\mathcal{H}$. Hence by [6, 4.25], every proper congruence on S is contained in $\mathcal{H}$.

(c) $\Rightarrow$ (d). The relation $\mathfrak{M}$ defined on S by

$$a \mathfrak{M} b \text{ if } axa \neq 0 \leftrightarrow bxb \neq 0 \text{ for all } x \in S$$

is by [7, 2.6] the largest 0-matrix congruence on S . Hence $\mathfrak{M}$ is proper and thus the hypothesis implies $\mathfrak{M} \subseteq \mathcal{H}$. Thus if $e, f \in E_S$ satisfy $e \mathfrak{M} f$ we obtain $e \mathcal{H} f$ so that $e = f$.

(d) $\Rightarrow$ (a). Suppose that, for all $\lambda \in \Lambda$, $p_{\lambda i} \neq 0 \Leftrightarrow p_{\lambda j} \neq 0$. There exists $\mu \in \Lambda$ such that $p_{\mu i} \neq 0$, so also $p_{\mu j} \neq 0$. Hence for any $(x; k, \lambda) \in S$,

$$\begin{aligned} (p_{\mu i}^{-1}; i, \mu)(x; k, \lambda)(p_{\mu i}^{-1}; i, \mu) \neq 0 &\Leftrightarrow p_{\mu k} \neq 0, p_{\lambda i} \neq 0 \Leftrightarrow p_{\mu k} \neq 0, p_{\lambda j} \neq 0 \\ &\Leftrightarrow (p_{\mu j}^{-1}; j, \mu)(x; k, \lambda)(p_{\mu j}^{-1}; j, \mu) \neq 0 \end{aligned}$$

which by the hypothesis implies $(p_{\mu i}^{-1}; i, \mu) = (p_{\mu j}^{-1}; j, \mu)$ and hence $i = j$. One shows analogously that $p_{\lambda i} \neq 0 \Leftrightarrow p_{\mu i} \neq 0$ for all $i \in I$ implies $\lambda = \mu$.

Even though the concept "no contractions" was defined for a sandwich matrix, the above proposition shows that for a completely 0-simple semigroup S , this notion does not depend on the particular Rees matrix representation. For this reason, we say that S has no contractions and, more generally, introduce the following concept.

Definition 6.3. A completely semisimple semigroup $S = S^0$ has no contractions if every nonzero principal factor of S is without contractions (i.e., satisfies any of the conditions in 6.2). If S is completely semisimple without zero, then S has no contractions if S^0 has none.

Proposition 6.4. Every completely semisimple semigroup S without contractions has the property that every congruence on S contained in $\mathcal{D}$ is also contained in $\mathcal{H}$.

Proof. Let σ be a congruence on S such that $\sigma \subseteq \mathcal{D}$, and let $P = J(a)/I(a)$ be a principal factor of S . If $I(a) = \emptyset$, then P is the kernel of S and the hypothesis implies that P must be a group, which then trivially implies that $\sigma|_P \subseteq \mathcal{H}|_P$. Suppose next that $I(a) \neq \emptyset$. Since $P^* = J(a) \setminus I(a)$, the restriction $\sigma|_{P^*}$ is an equivalence relation on P^* which can be extended to a congruence τ on P by letting the zero of P be τ -related only to itself. Hence τ is a proper congruence of the completely 0-simple semigroup P which by 6.2 implies that τ is contained in the $\mathcal{H}$ equivalence on P . Since the restriction of the $\mathcal{H}$ equivalence on S to P^* coincides with the restriction to P^* of the $\mathcal{H}$ equivalence on P , and P is an arbitrary principal factor of S , it follows that $\sigma \subseteq \mathcal{H}$, the $\mathcal{H}$ equivalence on S .

The converse of 6.4 holds if for every principal factor P of S , every proper congruence on P restricted to P^* can be extended to a congruence on S contained in $\mathcal{D}$. This occurs in the case we consider next. Recall that a semigroup $S = S^0$ is an orthogonal sum of (its sub-) semigroups S_α , $\alpha \in A$, if $S = \bigcup_{\alpha \in A} S_\alpha$, $S_\alpha S_\beta = S_\alpha \cap S_\beta = 0$ if $\alpha \neq \beta$ (0-direct union in [1, §6.3]); the semigroups S_α are called components. We will write $S = \sum_{\alpha \in A} \oplus S_\alpha$, and if there are only two components, $S = S_1 \oplus S_2$. A regular semigroup $S = S^0$ is called primitive if all its nonzero idempotents are primitive; these are precisely the semigroups which are orthogonal sums of completely 0-simple semigroups [7, 5.16].

Corollary 6.5. The following conditions on a primitive regular semigroup S are equivalent.

(a) S has no contractions.

(b) Every congruence on S which is contained in $\mathcal{D}$ is also contained in $\mathcal{H}$.

(c) For $e, f \in E_S$, $exe \neq 0 \Leftrightarrow fxf \neq 0$ for all $x \in S$ implies $e = f$.

Proof. (a) $\Rightarrow$ (b). This is a special case of 6.4.

(b) $\Rightarrow$ (c). Since S is an orthogonal sum of completely 0-simple semigroups, it follows without difficulty that the relation $\mathcal{M}$ defined in the proof of 6.2 (c) $\Rightarrow$ (d) is a congruence on S contained in $\mathcal{D}$ and thus also in $\mathcal{H}$, which implies that (c) holds.

(c) $\Rightarrow$ (a). Condition (c) then also holds in each completely 0-simple component of S and hence 6.2 implies (a).

Recall that an ideal I of a semigroup S is *prime* if for any $a, b \in S$, $aSb \subseteq I$ implies that either $a \in I$ or $b \in I$, and that I is a *matrix ideal* if it is both prime and categorical (see [7] where categorical is called quasi-complètement premier).

Lemma 6.6. Let $V = V^0$ be a semigroup categorical at zero with an ideal $S = \sum_{\alpha \in A} \bigoplus S_\alpha$ where each S_α is completely 0-simple. Then $V = (\sum_{\alpha \in A} \bigoplus T_\alpha) \oplus Q$ where, for every $\alpha \in A$, we have $S_\alpha \subseteq T_\alpha$ and 0 is a matrix ideal of T_α .

Proof. Let $T = \{v \in V \mid Sv \neq 0\} \cup 0$ and $Q = (V \setminus T) \cup 0$. If $s \in S$ and $v \in V$ are such that $sv \neq 0$, then $sv = (sv)u(sv)$ for some $u \in S$ since $sv \in S$ and S is regular. But then $us \in S$ and $v(us) \neq 0$. By symmetry, we conclude that $Sv \neq 0 \Leftrightarrow vS \neq 0$, and the definition of T is actually symmetric. If $t \in T$, $v \in V$, and $tv \neq 0$, then $st \neq 0$ for some $s \in S$ and hence the hypothesis implies that $stv \neq 0$, so that $tv, v \in T$. It follows similarly that $t \in T$, $v \in V$, and $vt \neq 0$ imply $vt, v \in T$. Consequently, T is an ideal of V and, for any $t \in T$, $q \in Q$, we have $tq = qt = 0$. If $x, y \in V$ and $xy \in T^*$, then $sxy \neq 0$ for some $s \in S$ and thus $x, y \in T$. It follows by contraposition that Q is a subsemigroup of V , which then shows that $V = T \oplus Q$, and by construction, $S \subseteq T$.

If $t \in T^*$, then $st \neq 0$ for some $s \in S$ and hence $st = (st)u(st)$ for some $u \in S$, so that $tTt \neq 0$. In such a case, 0 is called a *semiprime ideal* of T and since 0 is also a categorical ideal of T [7, 5.13] implies that $T = \sum_{\beta \in B} \bigoplus T_\beta$ where 0 is a matrix ideal of each T_β . It is easy to see that each S_α is an ideal of S and thus also of T . Since 0 is a prime ideal of each T_β , if $s_1 \in S_{\alpha_1}^*$, $s_2 \in S_{\alpha_2}^*$, and $S_{\alpha_1} \cup S_{\alpha_2} \subseteq T_\beta$, then for some $t \in T_\beta$, $s_1ts_2 \neq 0$. But $s_1ts_2 \in S_{\alpha_1}^* \cap S_{\alpha_2}^*$ and hence $\alpha_1 = \alpha_2$. Consequently, each T_β contains at most one S_α . If $t \in T_\beta^*$ then $st \in S$ for some $s \in S_\alpha^*$ for some α which shows that $S_\alpha \subseteq T_\beta$ and each T_β contains at least one S_α . Therefore we may label the T_β 's in such a way that S_α is contained only in T_α .

We will take for granted definitions and simple properties of an (ideal) extension, partial homomorphism, an extension determined by a partial homomorphism, and inflation (see [1, §4.4] and [3]). The next theorem and its corollaries were

established in [15] for the special case of a primitive inverse semigroup. It seems remarkable that in it and its corollaries the nature of the entire extension is dictated by its behavior at zero.

Theorem 6.7. *An extension V of a primitive regular semigroup S without contractions is determined by a partial homomorphism if and only if V has a categorical ideal I such that $I \cap S = 0$.*

Proof. Necessity. The proof is the same as the proof of necessity of [15, Theorem 1] and does not require the hypothesis of no contractions.

Sufficiency. As in the proof of sufficiency of [15, Theorem 1], the problem can be reduced to the case $I = 0$. By [7, 5.16] we can write $S = \sum_{\alpha \in A} \bigoplus S_\alpha$ where each S_α is completely 0-simple, so by 6.6 we obtain $V = (\sum_{\alpha \in A} \bigoplus T_\alpha) \oplus Q$ where, for each $\alpha \in A$, S_α is a 0-minimal ideal of T_α and 0 is a matrix ideal of T_α . Now the restriction of the smallest 0-matrix congruence Φ_α on T_α to S_α is a 0-matrix congruence on S_α . Since S_α has no contractions, 6.2 implies that the $\mathcal{H}$ equivalence $\mathcal{H}_\alpha$ on S_α is its only 0-matrix congruence, which then implies that $\Phi_\alpha|_{S_\alpha} = \mathcal{H}_\alpha$. Hence [7, 4.15.I] is satisfied and thus [7, 4.15.II] yields the existence of a homomorphism ϕ_α mapping T_α onto S_α and leaving the latter elementwise fixed. Defining ϕ on all of V by $v\phi = v\phi_\alpha$ if $v \in T_\alpha$ and $v\phi = 0$ if $v \in Q$, it follows at once that ϕ is a homomorphism of V onto S leaving the latter elementwise fixed. By [13, Proposition 2], V is an extension determined by a partial homomorphism (in fact, by $\phi|_{V \setminus S}$).

The proofs of the following corollaries are the same as those of the corollaries to [15, Theorem 1].

Corollary 6.8. *A semigroup V is an inflation of a primitive regular semigroup without contractions if and only if V is categorical at zero and V^2 is a primitive regular semigroup without contractions.*

Corollary 6.9. *Let S be a primitive regular semigroup without contractions and let T be a semigroup categorical at zero and having no nonzero proper ideals. Then an extension V of S by T is determined by a partial homomorphism if and only if V is categorical at zero.*

7. Completely semisimple semigroups without contractions. We will prove in this section that for this class of semigroups the converses of 5.3 and 5.4 are valid. These results have several interesting consequences. We begin by characterizing the property "no contractions" in terms of elements of a completely semisimple semigroup.

Proposition 7.1. *A completely semisimple semigroup S has no contractions if and only if, for any $e, f \in E_S$,*

$$e \in SexeS \Leftrightarrow f \in Sfx/S \text{ for all } x \in S \text{ implies } e = f.$$

Proof. Necessity. Let $e, f \in E_S$ and suppose that, for all $x \in S$, $e \in SexeS \Leftrightarrow f \in Sfx/S$. Since $e \in SeeeS$, we have $f \in Sfe/S \subseteq SeS = J(e)$ and hence $J(f) \subseteq J(e)$; by symmetry, we also have $J(e) \subseteq J(f)$. Letting $P = J(e)/I(e)$, we obtain $e, f \in P^*$. Let $x \in P$ and suppose that $exe \neq 0$ in P . Since P is completely (0-)simple, $e = uexv$ for some $u, v \in P^*$. Consequently, $e \in SexeS$ since $u, x, v \in S$ and the hypothesis implies that $f \in Sfx/S$, that is, $f = a/x/b$ for some $a, b \in S$. It follows that $f/x/f \in P^*$ which proves that $f/x/f \neq 0$ in P . By symmetry, we conclude that, for all $x \in P$, $exe \neq 0 \Leftrightarrow f/x/f \neq 0$ which by 6.2 yields $e = f$.

Sufficiency. First let P be a completely 0-simple semigroup and suppose that, for $e, f \in E_P$, we have $exe \neq 0 \Leftrightarrow f/x/f \neq 0$ for all $x \in P$. If $xey \neq 0$, then $xe \neq 0$ and $ey \neq 0$, so there exists $u \in P$ such that $eyux \neq 0$. The hypothesis then implies that $fyuxf \neq 0$ and thus $fy \neq 0$ and $xf \neq 0$ so that $xyf \neq 0$. By symmetry, we conclude that $xey \neq 0 \Leftrightarrow xfy \neq 0$ for all $x, y \in P$.

Now let P be a principal factor of S , let $e, f \in E_{P^*}$ and suppose that, for all $x \in P$, $exe \neq 0 \Leftrightarrow f/x/f \neq 0$. Suppose next that $e \in SeyeS$ for some $y \in S$. Then $e = aeyeb$ for some $a, b \in S$ and hence $ea, yebe \in P$ and $(ea)e(yebe) \neq 0$ in P . By the above, $(ea)f(yebe) \neq 0$ in P . Since $ea fy, be \in P$, again by the above, $(ea fy)f(be) \neq 0$ in P . Consequently $f \in P^* e a f y f b e P^* \subseteq S f y / S$; by symmetry, we conclude that for all $y \in S$, $e \in SeyeS \Leftrightarrow f \in S f y / S$, which by the hypothesis yields $e = f$. Therefore P has no contractions.

The next result provides a partial converse of 5.3.

Theorem 7.2. *The following conditions on a completely semisimple semigroup S without contractions are equivalent.*

- (a) S satisfies $\mathcal{D}$ -majorization.
- (b) All ideals of S are $\mathcal{D}$ -categorical.
- (c) S is a subdirect product of completely 0-simple semigroups without contractions.

Proof. (a) $\Rightarrow$ (c). As we have seen in the proof of 3.4, S is a subdirect product of its principal factors, which here by the overall hypothesis have no contractions.

(c) $\Rightarrow$ (a). This is a part of 3.4 and holds in any completely semisimple semigroup.

(a) $\Rightarrow$ (b). This is a special case of 5.3.

(b) $\Rightarrow$ (a). Let $e, f, g \in E_S$ and suppose that $e \geq f$, $e \geq g$, $f \mathcal{D} g$. Let $P = J(f)/I(f)$ and suppose that $f/x/f \neq 0$ in P for some $x \in P$. Then $fexf \neq 0$ and hence $ex \in P^*$. Consequently, $ge, ex \notin I(f)$ and $ge \mathcal{D} ex$ and thus the hypothesis implies that $gex \notin I(f)$ (if $I(f) = \emptyset$, this is trivially satisfied). Hence in P , gx

$\neq 0$, and an analogous argument shows that also $xg \neq 0$. But then $g x g \neq 0$; by symmetry, we conclude that, for all $x \in P$, $f x f \neq 0 \Leftrightarrow g x g \neq 0$. Since P has no contractions, it follows that $f = g$.

The next corollary should be compared with 3.4.

Corollary 7.3. *The following conditions on a semigroup S are equivalent.*

- (a) S is a completely semisimple inverse semigroup satisfying $\mathcal{D}$ -majorization.
- (b) S is a completely semisimple inverse semigroup all of whose ideals are $\mathcal{D}$ -categorical.
- (c) S is regular and a subdirect product of Brandt semigroups.

The next theorem is a partial converse of 5.4.

Theorem 7.4. *In a completely semisimple semigroup S without contractions all of whose ideals are categorical, E_S is a tree.*

Proof. Let $e, f, g \in E_S$ and suppose that $e \geq f$, $e \geq g$. If $J(f) \neq J(fg) \neq J(g)$, then $fe, eg \notin J(fg)$ and the hypothesis implies that $fg = feg \notin J(fg)$, a contradiction. By symmetry, we may suppose that $J(g) = J(fg)$ so that $J(g) \subseteq J(f)$. If $I(g) = \emptyset$, then the hypothesis implies that $J(g) = J_g$ is a group, and $J(g) \subseteq J(f)$ by 2.11 yields $f \geq g$ since g is the only idempotent in J_g .

Suppose next that $I(g) \neq \emptyset$. Since $I(g)$ is a categorical ideal in S , we obtain that the semigroup $V = J(e)/I(g)$ is categorical at zero. Further, $T = J(g)/I(g)$ is a completely 0-simple semigroup without contractions and an ideal of V . Letting $I = 0$ in 6.7, we conclude that V is an extension of T determined by a partial homomorphism ϕ . Hence the hypothesis $e \geq g$ implies that $e\phi \geq g$, but since T is completely 0-simple, we must have $e\phi = g$. Again by 2.11, the hypothesis $J(g) \subseteq J(f)$ implies the existence of $b \in E_{J_g}$ such that $f \geq b$. But then $e \geq f \geq b$ which as above implies that $e\phi = b$. Consequently, $f \geq b = e\phi = g$ as desired.

Corollary 7.5. *In a completely semisimple semigroup S without contractions, E_S is a tree if and only if all ideals of S are categorical.*

In particular, the conclusion is valid in any completely semisimple inverse semigroup; this is precisely the content of [15, Theorem 2].

Proposition 7.6. *If S is a regular semigroup in which $\mathcal{D} = \mathcal{J}$ and all of whose ideals are categorical, then $S/\mathcal{J}$ is a tree.*

Proof. Suppose that $J(a) \supseteq J(b)$ and $J(a) \supseteq J(c)$. By 2.11, there exist $f \in E_{J_b}$ and $g \in E_{J_c}$ such that $e \geq f$ and $e \geq g$. Since then $feg = fg \in J(fg)$, the hypothesis implies that either $f \in J(fg)$ or $g \in J(fg)$. Consequently, either $J(f) = J(fg)$ or $J(g) = J(fg)$, so that either $J(b) \subseteq J(c)$ or $J(c) \subseteq J(b)$ and thus $S/\mathcal{J}$ is a tree.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vols. I, II, Math. Surveys, vol. 7, Amer. Math. Soc., Providence, R. I., 1961, 1967. MR 24 #A2627; MR 36 #1558.
2. N. Farès, *Idempotents et $\mathcal{D}$ -classes dans les demi-groupes et les anneaux*, C. R. Acad. Sci. Paris Sér. A-B 269 (1969), A341-A343. MR 40 #2777.
3. P. A. Grillet and M. Petrich, *Ideal extensions of semigroups*, Pacific J. Math. 26 (1968), 493-508. MR 38 #5967.
4. N. Kimura, *Note on idempotent semigroups*. III, Proc. Japan Acad. 34 (1958), 113-114. MR 20 #4604.
5. B. P. Kočin, *The structure of inverse ideal-simple ω -semigroups*, Vestnik Leningrad. Univ. 23 (1968), no. 7, 41-50. (Russian) MR 37 #2881.
6. G. Lallement, *Demi-groupes réguliers*, Ann. Mat. Pura Appl. (4) 77 (1967), 47-129. MR 37 #1505.
7. G. Lallement and M. Petrich, *Décompositions I -matricielles d'un demi-groupe*, J. Math. Pures Appl. (9) 45 (1966), 67-117. MR 33 #5765.
8. ———, *Structure d'une classe de demi-groupes réguliers*, J. Math. Pures Appl. (9) 48 (1969), 345-397. MR 41 #3638.
9. E. S. Ljapin, *Semigroups*, Fizmatgiz, Moscow, 1960; English transl., Transl. Math. Monographs, vol. 3, Amer. Math. Soc., Providence, R. I., 1968. MR 22 #11054.
10. W. D. Munn, *Regular ω -semigroups*, Glasgow Math. J. 9 (1968), 46-66. MR 37 #5316.
11. M. Petrich, *Sur certaines classes de demi-groupes*. I, Acad. Roy. Belg. Bull. Cl. Sci. (5) 49 (1963), 785-798. MR 29 #3560.
12. ———, *Kongruenzen einiger spezieller vollständig einfacher Halbgruppen*, Math. Nachr. 32 (1966), 73-81. MR 35 #272.
13. ———, *On extensions of semigroups determined by partial homomorphisms*, Nederl. Akad. Wetensch. Proc. Ser. A 69 = Indag. Math. 28 (1966), 49-51. MR 33 #5775.
14. ———, *Representations of semigroups and the translational hull of a regular Rees matrix semigroup*, Trans. Amer. Math. Soc. 143 (1969), 303-318. MR 40 #253.
15. ———, *On a class of completely semisimple inverse semigroups*, Proc. Amer. Math. Soc. 24 (1970), 671-676. MR 41 #372.
16. M. Yamada, *Regular semigroups whose idempotents satisfy permutation identities*, Pacific J. Math. 21 (1967), 371-392. MR 37 #2887.
17. M. Yamada and N. Kimura, *Note on idempotent semigroups*. II, Proc. Japan Acad. 34 (1958), 110-112. MR 20 #4603.

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